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An Extended Extremal Approach to the Definition of the Nonlocal Photon Form Factors

E. PAPP

Polytechnic Institute of Cluj, Physics Department, 3400-*Cluj, Romania*

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Abstract

Proofs have been given that the extremal technique to define the photon form factors of the nonlocal quantum electrodynamics needs some extensions. There is evidence that the use of form factors that are not entire analytical functions does not have to be rejected from the very beginning. In this respect several mutually connected form factors have been calculated and discussed. The short-distance behavior of the electrostatic potential is then established.

1. *Introduction*

Attempts have been made to define the bounded value of the electrostatic potential at small distances both in classical (Born, 1934) and quantum physics (Wataghin, 1934). Among the methods proposed in this respect a special interest is raised by the nonlocal approach to the quantum electrodynamics (Yukawa, 1950; Blokhintsev, 1957; Efimov, 1972; Marnelius, 1974). The present day high-energy collision processes show that neither the existence of the particle structure nor the nonzero particle radius can be ignored. In this letter sense the existence of the high-energy impact factors is significant (Cheng and Wu, 1973). The above-mentioned difficulties can be—at least in principle—satisfactorily overcome within the nonlocal approach to the electromagnetic interaction. The formulation of the nonlocal theory needs by itself certain extensions of the standard (local) formalism. To formulate the nonlocal theory we have to require that the local theory be the limiting case of the nonlocal one. In this line the main object of the present paper is to analyze the physical meaning of some mutually correlated photon form factors. For this purpose we shall adequately reformulate certain aspects of the nonlocal description of the Coulombian interaction proposed by Efimov (1972, 1974). Such a reinterpretation is of real interest since it is able to show the meaning and role of the photon form factors in the description of the particle structure.

In Section 2 there are presented some preliminaries and notation. The extremal approach to the definition of the photon form factor is analyzed in Section 3, thus also proving that there are needed certain extensions and refinments of the initial formalism. Proofs are given that the extremal approach does not necessarily possess a single solution. An alternative approach to the initial extremal problem is proposed in Section 4. The formulation proposed possesses the property of taking as finite the main functionals of the theory. The adequacy of an additional form factor is discussed in Section 6. The main integrals used are presented in the Appendix.

2. *Preliminaries and Notation*

The nonlocal approach to the Coulombian interaction leads to the potential (Efimov, 1972)

$$W(r) = \frac{e}{2\pi^2} \int_r^\infty \frac{dr'}{r'} \frac{1}{\sqrt{r'^2 - r^2}} U\left(\frac{r'^2}{l^2}\right) \tag{2.1}$$

where l is the fundamental length and where

$$U\left(\frac{r^2}{l^2}\right) = r \int_0^\infty d\, kJ_1(kr) V(k^2 l^2) \tag{2.2}$$

The photon form factor has been assumed to be an entire analytical function of the order $\frac{1}{2} \leqslant \rho < \infty$, which takes positive values for the real u values and which fulfills the conditions

$$V^*(u) = V(u^*), \qquad V(0) = 1 \tag{2.3}$$

and

$$V(u) = O(1/u^\alpha), \qquad u \to \infty, \qquad (\alpha > 0) \tag{2.4}$$

In these conditions the possibility exists of defining the finite value of the Coulombian potential and of the electron self-energy too.

The square of the electron charge radius is given by

$$\langle r^2 \rangle = -4l^2 V'(0) \tag{2.5}$$

so that in order to assure the finite nonzero value of the electron charge radius the condition

$$-\infty < V'(0) < 0 \tag{2.6}$$

has been imposed. However, the limiting case of the zero charge radius has to be included in the present theory, too. Consequently, we shall extend the condition (2.6) as

$$-\infty < V'(0) \leqslant 0 \tag{2.7}$$

The short-distance behavior of the coulombian potential and of the function $(1/r^2)U(r^2/l^2)$ is given by the functions

$$W^{(0)}[V] \equiv \lim_{r\to 0} W(r) = \frac{e}{4\pi^2 l} \int_0^\infty du \frac{V(u)}{\sqrt{u}} \tag{2.8}$$

and

$$D^{(0)}[V] \equiv \lim_{r\to 0} \frac{1}{r^2} U\left(\frac{r^2}{l^2}\right) = \frac{1}{4l^2} \int_0^\infty du\ V(u) \tag{2.9}$$

respectively. The modified photon propagator $D(r^2)$ takes the maximum value at the origin:

$$\max D(r^2) = D(0) = \frac{1}{16\pi^2 l^2} \int_0^\infty du\ V(u) \tag{2.10}$$

In these conditions we shall additionally require the physical form factor to take as finite both the functions $W^{(0)}[V]$ and $D^{(0)}[V]$, too.

The existence of the particle structure need not be exlusively expressed by the non zero value of the charge-radius or by the existence of a charge distribution. Indeed, the electron radius $e^2/2m_0c^2$ is not only a charge radius it also expresses the lower limit of the measurable quantum mechanical high-energy space shift (Papp, 1976). Moreover, the electronic and protonic impact factors (which establish—up to a certain factor—the value of the high-energy scattering amplitude) are given by $e^2/2m_0c^2$ and $e^2/2M_0c^2$, respectively (Cheng and Wu, 1969). In this context we shall consider that the requirement that the functions $W^{(0)}[V]$ and $D^{(0)}[V]$ take finite values implicitly contains elements of the high-energy space-time description, too.

3. *The External Approach*

In the absence of another general method an extremal approach has been formulated to choose explicitly the form of the photon form factor (Efimov, 1972, 1974). For this purpose it has been initially required that the form-factor take the extremuum of the function $W^{(0)}[V]$; thus we obtain the evaluation

$$V_1(u) = \frac{\sin\sqrt{u}}{\sqrt{u}}^{\,2} \tag{3.1}$$

Consequently

$$W^{(0)}[V_1] = e/4\pi l \tag{3.2}$$

and

$$\langle r^2 \rangle^{(1)} = \tfrac{4}{3} l^2 \tag{3.3}$$

so that a well-defined physical meaning can be attributed to the fundamental space constant l. However, the above expression of the form-factor is not a satisfactory one since [see equation (A1) in the Appendix]

$$D^{(0)}[V_1] = \infty \tag{3.4}$$

so that the photon propagator ceases—contrary to the initial aim—to be bounded at the origin.

It can easily be seen that the extremal problem does not necessarily possess a single solution. Indeed, performing the Fourier transform $\tilde{V}(v)$ of the above form factor and requiring both the functions $V(u)$ and $\tilde{V}(v)$ to take real values for the real values of their arguments, we have

$$V(u) = 2\int_0^\infty dv\, \cos uv\, \tilde{V}(v) \tag{3.5}$$

Consequently [see equation (A2)]

$$\int_0^\infty \frac{du}{\sqrt{u}}\, V(u) = \sqrt{2\pi}\int_0^\infty \frac{du}{\sqrt{u}}\, \tilde{V}(u) \tag{3.6}$$

This result shows that the function $W^{(0)}[V]$ takes the extremal value with respect to both the functions $V_1(u)$ and $\tilde{V}_1(u)$, respectively. Inverting the relation (3.5) one obtains

$$\tilde{V}_1(u) = \frac{2}{\pi}\int_0^\infty dv\, \frac{1}{v}\sin^2 v\, \cos uv^2 \tag{3.7}$$

However, the second solution has to be rejected as $\tilde{V}_1(0) = \infty$. The above results let us conclude that the initial formulation of the extremal problem requires some additional refinements.

4. *The Another Extremal Approach*

In agreement with the tasks we have proposed, a good physical form factor has to take finite the essential quantities of the theory and especially the functions $W^{(0)}[V]$ and $D^{(0)}[V]$. For this purpose it is quite natural to require the photon propagator (at the origin) to be an extremuum. Expressing the function $D^{(0)}[V]$ as

$$D^{(0)}[V] = \frac{1}{2l^2}\int_0^\infty \frac{du}{\sqrt{u}}\, V(\sqrt{u}) = \frac{1}{l^2}\int_0^\infty \frac{du}{\sqrt{u}}\, V(2\sqrt{u}) \tag{4.1}$$

and setting

$$V_2(2\sqrt{u}) = V_1(u) \tag{4.2}$$

it can be remarked that the present extremal problem is reduced to the previous one. Consequently the new form factor is

$$V_2(u) = \frac{4}{u^2}\sin^2\frac{u}{2} \tag{4.3}$$

so that the extremal value of the function $D^{(0)}[V]$ is given by [see equation (A3)]

$$D^{(0)}[V_2] = \pi/4l^2 \tag{4.4}$$

Performing the calculations one obtains [equation (A4)]

$$W^{(0)}[V_2] = \frac{\sqrt{2\pi}}{3}\frac{e}{\pi^2 l} \simeq 0.95\frac{e}{4\pi l} \tag{4.5}$$

so that the upper bound of the Coulombian potential is practically identical to the previous expression (3.2). The form-factor so obtained is an entire analytical function of the order $\rho = 1$, which takes positive values for the real u values and which satisfies the conditions (2.3) and (2.4).

Using the Fourier transform of the form factor $V_2(u)$, we have [equation (A5)]

$$U_2\left(\frac{r^2}{l^2}\right) = 4\int_1^\infty \frac{du}{u^2}\left(1 - \frac{1}{u}\right)\sin^2\frac{r^2}{8l^2}u \tag{4.6}$$

so that [equations (A6) and (A7)]

$$W_2(r) = \frac{e}{\pi r}\int_1^\infty \frac{du}{u^3}\left(1 - \frac{1}{u^2}\right)\left[S\left(\frac{r}{2l}u\right) + C\left(\frac{r}{2l}u\right)\right] \tag{4.7}$$

where the behavior of the Fresnel (sine and cosine integral) functions at the origin and infinity is well established.

In the present case we have

$$V_2'(0) = 0 \tag{4.8}$$

so that the physical meaning of the fundamental space constant l has now to be established only in terms of the finite evaluations (4.4) and (4.5). We can thus consider–in agreement with the previous remarks–that the nonzero charge radius is not the only possibility to express the particle structure.

It can be easily verified that

$$\int_0^\infty du\, V_2(u) = \pi\int_0^\infty du\, \Phi^2\left(\frac{u}{2}\right) \tag{4.9}$$

where [equation (A8)]

$$\Phi(u) = \theta(1 - u), \qquad u \in (0, \infty) \tag{4.10}$$

is the Fourier transform of the function $[V_2(u)]^{1/2}$. Consequently, the other solution of the extremal problem is the distribution

$$V_3(u) = \theta^2(1 - |u|), \qquad u \in (-\infty, +\infty) \tag{4.11}$$

The function $U(r^2/l^2)$ takes correspondingly the form [equation (A9)]

$$U_3\left(\frac{r^2}{l^2}\right) = 1 - J_0\left(\frac{r}{l}\right) \tag{4.12}$$

which allows the form factor $V_3(u)$ to be effectively replaced by the distribution

$$V_3^{(e)}(u) = \begin{cases} 1, & |u| \leqslant 1 \\ 0, & |u| > 1 \end{cases} \tag{4.13}$$

The potential corresponding to this distribution is

$$W_3(r) = W^{(a)}(r) + W_3^{(1)}(r) \tag{4.14}$$

where

$$W^{(a)}(r) = e/4\pi r \tag{4.15}$$

is the Coulombian potential and [equation (A10)]

$$W_3^{(1)}(r) = -\frac{er}{16\pi\sqrt{\pi}\, l^2} G_{13}^{20}\left(\frac{r^2}{4l^2}\middle|\begin{matrix}0\\ -1/2, -1, -1\end{matrix}\right) \tag{4.16}$$

is the regularizing potential implied by the Bessel function $J_0(r/l)$. The Meyer function can be expressed as

$$G_{13}^{20}\left(x\middle|\begin{matrix}0\\ -1/2, -1, -1\end{matrix}\right) = -\sum_{l=0}^{\infty} \frac{(-1)^l 2^{l+2} x^{l-1/2}}{\sqrt{\pi}\, l!(2l+1)^2(2l-1)!!} + \frac{\sqrt{\pi}}{x} \tag{4.17}$$

so that

$$\lim_{x\to 0} G_{13}^{20}\left(x\middle|\begin{matrix}0\\ -1/2, -1, -1\end{matrix}\right) = \frac{\sqrt{\pi}}{x} - \frac{4}{\sqrt{\pi x}} \tag{4.18}$$

Consequently

$$\lim_{r\to 0} W_3^{(1)}(r) = \frac{e}{2\pi^2 l} - \frac{e}{4\pi r} \tag{4.19}$$

In these conditions both the functionals $W^{(0)}[V]$ and $D^{(0)}[V]$ take the finite values

$$W^{(0)}[V_3] = e/2\pi^2 l \tag{4.20}$$

and

$$D^{(0)}[V_3] = 1/4l^2 \tag{4.21}$$

The dominant contribution to the potential $W_3(r)$ for large distances is given by $W^{(a)}(r)$ because the Bessel function $J_0(r/l)$ vanishes asymptotically.

Both the distributions $V_3(u)$ and $V_3^{(e)}(u)$ are not entire analytical functions, but they give raise to physically meaningful results. On the other hand these functions do not imply additional singularities in the propagator, so that they can be admitted in this respect, too. Moreover, the above functions also express in fact the usual method of performing the momentum cutoff. In these conditions these functions preserve a justified role as candidate form factors, since they also fulfill all the (other) previously formulated requirements.

5. *The Correlated Form Factor*

Similarly to the photon form factor $V_2(u)$, the form factor $V_3(u)$ also gives rise to a zero charge radius. It can be easily seen that the absence of the linear term in the u variable implies the vanishing of the form-factor derivatives in the origin. Indeed,

$$V_2(u) \underset{u\to 0}{\simeq} 1 - u^2/12, \qquad V_3(u) \underset{u\to 0}{=} 1 \tag{5.1}$$

whereas

$$V_1(u) \underset{u\to 0}{\simeq} 1 - u/3 \tag{5.2}$$

It can be also noticed that the form factor $V_1(u)$ takes appreciable values only in the vicinity of the origin:

$$\sqrt{|u|} \lesssim \pi/2 \tag{5.3}$$

Gathering the above remarks we can propose qualitatively as a suitable form factor the function

$$V(u) = \begin{cases} 1 - (1/M^2)|u|, & |u| \leqslant 1 \\ 0, & |u| > 1 \end{cases} \tag{5.4}$$

where M is a real parameter. But this result agrees with the one that can be obtained if we would define a correlated solution to the extremal approach formulated in Section 4. Indeed, the solution $V_2(u)$ implies—with respect to the relation (3.6)—the Fourier transform

$$V_4(u) \equiv \tilde{V}_2(u) = \begin{cases} 1 - |u|, & |u| \leqslant 1 \\ 0, & |u| > 1 \end{cases} \tag{5.5}$$

as a (connected) solution of the extremal problem, too. In this latter case we have [equation (A11)]

$$U_4\left(\frac{r^2}{l^2}\right) = 1 - J_0\left(\frac{r}{l}\right) - J_2\left(\frac{r}{l}\right) \tag{5.6}$$

The corresponding potential takes the form

$$W_4(r) = W^{(a)}(r) + W_3^{(1)}(r) + W_4^{(1)}(r) \tag{5.7}$$

where [equation (A12)]

$$W_4^{(1)}(r) = -\frac{el^2}{2\pi^2}\left(\frac{l}{r}\sin\frac{r}{l} - \cos\frac{r}{l}\right) \tag{5.8}$$

so that a certain (not too large) modification of the previous potential $W_3(r)$ is implied especially at short distances. Consequently

$$W^{(0)}[V_4] = \frac{e}{3\pi^2 l} \simeq \frac{1}{2.35} W^{(0)}[V_1], \qquad D^{(0)}[V_4] = \frac{1}{8l^2} \tag{5.9}$$

whereas

$$\langle r^2 \rangle^{(4)} = 4l^2 \tag{5.10}$$

thus also obtaining the present charge radius as the double of the fundamental length. It can be also mentioned that—in a larger sense—the existence of the elementary length $2l$ is "binarily" equivalent to the existence of the length l (Papp, 1973). The above results show the usefulness of the form factor $V_4(u)$ to describe, in a certain mutual connection, both the quantum mechanical and the properly (electromagnetic) charge-structure aspects of the electron structure. In this respect the existence of the form factor $V_4(u)$, which makes finite the main functions of the theory, is of special physical interest. The potential energy becomes—with respect to the form factor $V_2(u)$—identical to the electron rest mass when

$$l = \frac{e^2}{4\pi m_0 c^2} \tag{5.11}$$

In the present case the electron rest mass is approximately 2.35 times larger than the potential energy. This fact would mean that in the latter case there is an essential contribution of the "nonelectromagnetic" structure effects.

6. *Conclusions*

In this paper we have restricted ourselves to the analysis of the short-distance behavior of the Coulombian potential in the nonlocal quantum electrodynamics. Proofs have been given that the possibility of using form-factors other than the entire analytical functions is of real physical interest. The so obtained form factors are able to express certain aspects of the electron structure. The analyzed form factors are in fact mathematically connected functions. It can be assumed that a rather complete description of the set of the physical form-factors so obtained needs a subsequent generalization: The photon form factors have to result from the general principles of the short-length measurements, too. However, as has been proved, significant results can be obtained by an adequate extension of this technique. From the physical point of view the analyzed form factors differ in the extent they engage the fundamental space constant l. The existence of a set of (mutually connected) electron lengths, which do not relatively take too different values, is in agreement with the binary description of the length measurements (Papp, 1973). In this way a mutual connection can be qualitatively established between the high-energy binary space-time description and the nonlocal theory, too. In this respect the fundamental space constant l has to be interpreted as the elementary space imprecision.

Appendix

The main integrals used are

$$\int_0^\infty \frac{dx}{x} \sin^2 x = \infty \tag{A1}$$

$$\int_0^\infty \frac{dx}{\sqrt{x}} \cos ax = \sqrt{\frac{\pi}{2a}}, \qquad a > 0 \tag{A2}$$

$$\int_0^\infty \frac{dx}{x^2} \sin^2 ax = \frac{\pi}{2} a \tag{A3}$$

$$\int_0^\infty \frac{dx}{x^2\sqrt{x}} \sin^2 \frac{x}{2} = \frac{\sqrt{2\pi}}{3} \tag{A4}$$

$$\int_0^\infty \frac{dx}{\sqrt{x}} \cos bx J_1(a\sqrt{x}) = \sqrt{\frac{\pi}{b}} J_{1/2}\left(\frac{a^2}{8b}\right) \sin^2 \frac{a^2}{8b}, \qquad b > 0 \tag{A5}$$

$$\int_1^\infty \frac{dx}{\sqrt{x-1}} \sin ax = \sqrt{\frac{\pi}{2a}} (\sin a + \cos a) \tag{A6}$$

$$\int_0^1 dx \sin ax^2 = \sqrt{\frac{\pi}{2a}} S(\sqrt{a}), \qquad \int_0^1 dx \cos ax^2 = \sqrt{\frac{\pi}{2a}} C(\sqrt{a}) \tag{A7}$$

$$\int_0^\infty \frac{dx}{x} \sin ax \cos bx = \frac{\pi}{2} \theta(a - b) \tag{A8}$$

$$\int_0^a dx J_1(x) = 1 - J_0(a) \tag{A9}$$

$$\int_1^\infty \frac{dx}{x\sqrt{x(x-1)}} J_1(a\sqrt{x}) = \frac{\sqrt{\pi}}{8} a^3 G_{13}^{20}\left(\frac{a^2}{4}\middle|{}^{0}_{-1/2,-1,-1}\right) \tag{A10}$$

$$\int_0^1 dx\, x^2 J_1(ax) = a^{-1} J_2(a) \tag{A11}$$

$$\int_1^\infty \frac{dx}{x\sqrt{x-1}} J_2(a\sqrt{x}) = \sqrt{\frac{2\pi}{a}} J_{3/2}(a) \tag{A12}$$

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